Constitutive sets of convex static systems

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Abstract. The principle of virtual work for dissipative systems is stated. Partially controlled systems are discussed and the concept of a generating families of forms is introduced. The notion of a critical point of a family of convex forms is introduced and discussed. A number of examples is given.

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1. Introduction.

The known procedures of generating the dynamics of a conservative mechanical system from the action functional, the Lagrangian or the Hamiltonian do not apply directly to dissipative systems. In preparation for a variational formulation of dynamics with dissipation we are adapting in the present note concepts borrowed from convex analysis to differential geometric setting. We are applying these concepts to examples of static systems. Holonomic and non holonomic constraints as well as partial control are considered. We believe that the principle of virtual work applied to static systems is a simple fundamental model of all variational principles of classical physics.

The paper is organized as follows. In Section 2 we state the principle of virtual work which is incorporated in the definition of the constitutive set. In Sections 3-6 we adapt concepts of convex analysis to static systems. In Sections 7-8 we consider the case of partially controlled potential systems. A generalization to the case of dissipative systems leads to the notion of generating families of forms and their critical points (Sections 9-10).

Propositions and theorems stated in this note without proofs are found in [2].

2. The principle of virtual work and examples of static systems.

The *principle of virtual work* well known in statics of mechanical systems is a master model for all variational principles of classical physics. We state a simple version of the principle. The *configuration space* of a static system is a set Q assumed here to be a differential manifold. The system is represented by its *internal energy*

$$U : C^0 \to \mathbb{R} \tag{1}$$

defined on a *constraint set* $C^0 \subset Q$ assumed to be a submanifold. The *principle of virtual work* is incorporated in the definition

$$S = \left\{ f \in \mathsf{T}^*Q \; ; \; q = \pi_Q(f) \in C^0, \; \forall_{\delta q \in \mathsf{T}_q C^0} \; \langle \mathrm{d}U, \delta q \rangle = \langle f, \delta q \rangle \right\} \tag{2}$$

of the *constitutive set*. The constitutive set characterizes the response of a static system to control by external forces.

The configuration spaces in the examples of static systems in the present section are constructed from the Euclidean affine space M of three dimensions modeled on a vector space V. The Euclidean metric is represented by a metric tensor $g : V \to V^*$.

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Example 1. Let the configuration space of a material point be the affine space $Q = M$. The space $\delta Q = V$ is the model space of Q and $F = V^*$ is the dual of the model space. The point with configuration $q \in Q$ is tied with a rigid rod of length a to a fixed point with configuration q_0. The configuration q is constrained to the sphere

$$C^0 = \{q \in Q\ ;\ \|q - q_0\| = a\}\,. \tag{3}$$

With the internal energy $U = 0$ the constitutive set is the set

$$S = \left\{(q, f) \in Q \times F\ ;\ \|q - q_0\| = a,\ f = a^{-2}\langle f, q - q_0\rangle g(q - q_0)\right\}. \tag{4}$$

▲

A more general static system is characterized by *virtual work function*

$$\sigma : C^1 \to \mathbb{R} \tag{5}$$

defined on a *constraint set* $C^1 \subset \mathsf{T}Q$. For each $q \in C^0 = \tau_Q(C^1)$ the set $C^1_q = C^1 \cap \mathsf{T}_qQ$ is a cone in the sense that if $\delta q \in C^1_q$, then $\lambda\delta q \in C^1_q$ for each $\lambda \geqslant 0$. The virtual work function is homogeneous in the sense that $\sigma(\lambda\delta q) = \lambda\sigma(\delta q)$ if $\lambda \geqslant 0$. The inclusion $C^1 \subset \mathsf{T}C^0$ is usually verified. Constraints are said to be *holonomic* if $C^1 = \mathsf{T}C^0$. The *tangent set* $\mathsf{T}C$ of any set $C \subset Q$ is defined by

$$\mathsf{T}C = \{\delta q \in \mathsf{T}Q\ ;\ \text{there is a curve }\ \gamma : \mathbb{R} \to Q \text{ such that } \mathsf{t}\gamma(0) = \delta q \text{ and } \gamma(s) \in C \text{ if } s \geqslant 0\} \tag{6}$$

Holonomic constraints are fully represented by the constraint set C^0. Constraints are said to be *bilateral* at $q \in C^0$ if $\delta q \in C^1_q$ implies that $-\delta q \in C^1_q$. If constraints are not bilateral at $q \in C^0$, then they are said to be *unilateral*. Constraints in the above example are holonomic and bilateral.

A more general version of the *principle of virtual work* is incorporated in the definition

$$S = \left\{f \in \mathsf{T}^*Q\ ;\ q = \pi_Q(f) \in C^0,\ \forall_{\delta q \in C^1_q}\ \sigma(\delta q) \geqslant \langle f, \delta q\rangle\right\} \tag{7}$$

of the *constitutive set*.

Example 2. Let the configuration space of a material point be the affine space $Q = M$. We use definitions and symbols introduced in Example 1. The material point is not constrained and is subject to isotropic static friction. The virtual work is the function

$$\sigma : Q \times \delta Q \to \mathbb{R} : (q, \delta q) \mapsto \rho(q)\|\delta q\| = \rho(q)\sqrt{\langle g(\delta q), \delta q\rangle}. \tag{8}$$

The set

$$S = \left\{(q, f) \in Q \times F\ ;\ \forall_{\delta q \in \delta Q}\ \rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle\right\} \tag{9}$$

is the constitutive set.

Let $(q, f) \in S$. By setting $\delta q = g^{-1}(f)$ in the inequality

$$\rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle \tag{10}$$

we obtain the inequality

$$\rho(q)\|f\| \geqslant \|f\|^2. \tag{11}$$

Hence,

$$S \subset \{(q, f) \in Q \times F\ ;\ \|f\| \leqslant \rho(q)\}\,. \tag{12}$$

Let (q, f) satisfy the inequality

$$\|f\| \leqslant \rho(q). \tag{13}$$

The relation

$$\langle f, \delta q\rangle \leqslant |\langle f, \delta q\rangle| \leqslant \|f\|\|\delta q\| \leqslant \rho(q)\|\delta q\| \tag{14}$$

is derived from the Schwarz inequality

$$|\langle f, \delta q\rangle| \leqslant \|f\|\|\delta q\|. \tag{15}$$

We have shown that

$$S = \{(q, f) \in Q \times F\ ;\ \|f\| \leqslant \rho(q)\}\,. \tag{16}$$

▲

Passages similar to that from (9) to (16) will appear in several examples.

Example 3. The configuration space of a skate is the set $Q = M \times D$, where D is the projective space of directions in M. We use the Euclidean metric in M to identify the space D with the unit circle

$$D = \{\vartheta \in V \; ; \; \langle g(\vartheta), \vartheta \rangle = 1\} . \tag{17}$$

Virtual displacements are elements of the space $\mathsf{T}Q = M \times V \times \mathsf{T}D$, where

$$\mathsf{T}D = \{(\vartheta, \delta\vartheta) \in D \times V \; ; \; \langle g(\vartheta), \delta\vartheta \rangle = 0\} . \tag{18}$$

The skate is a system with non holonomic constraints. The set C^0 is the entire space Q. The constraint consists in restricting virtual displacements in M to those parallel to the direction specified by an element of D. Thus

$$C^1 = \big\{(x, \delta x, \vartheta, \delta\vartheta) \in \mathsf{T}Q \; ; \; \exists_{\lambda\in\mathbb{R}} \; \delta x = \lambda\vartheta\big\} . \tag{19}$$

The constitutive set is a subset of the space $\mathsf{T}^*Q = M \times V^* \times \mathsf{T}^*D$, where

$$\mathsf{T}^*D = \big\{(\vartheta, \tau) \in D \times V^* \; ; \; \langle \tau, \vartheta \rangle = 0\big\} . \tag{20}$$

is the space chosen as the dual of $\mathsf{T}D$.

Let the skate be subject to friction represented by a non negative function $\rho : Q \to \mathbb{R}$. The virtual work is the function

$$\sigma : C^1 \to \mathbb{R} : (x, \delta x, \vartheta, \delta\vartheta) \mapsto \rho(x, \vartheta)\|\delta x\| = \rho(x, \vartheta)\sqrt{\langle g(\delta x), \delta x \rangle}. \tag{21}$$

The set

$$S = \Big\{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q \; ; \; \forall_{(x,\delta x,\vartheta,\delta\vartheta)\in C^1} \; \rho(x, \vartheta)\|\delta x\| \geqslant \langle f, \delta x \rangle + \langle \tau, \delta\vartheta \rangle\Big\} \tag{22}$$

is the constitutive set. The equality $\tau = 0$ is obtained by setting $\delta x = 0$ in the inequality

$$\rho(x, \vartheta)\|\delta x\| \geqslant \langle f, \delta x \rangle + \langle \tau, \delta\vartheta \rangle \tag{23}$$

with arbitrary $\delta\vartheta$. By setting $\delta x = \lambda\vartheta$ we arrive at the inequality

$$\rho(x, \vartheta)|\lambda| \geqslant \lambda\langle f, \vartheta \rangle \tag{24}$$

for each $\lambda \in \mathbb{R}$. The inequality must be satisfied for $\lambda = \langle f, \vartheta \rangle$. Hence $\rho(x, \vartheta)|\langle f, \vartheta \rangle| \geqslant \langle f, \vartheta \rangle^2$ and $|\langle f, \vartheta \rangle| \leqslant \rho(x, \vartheta)$. If $|\langle f, \vartheta \rangle| \leqslant \rho(x, \vartheta)$, then

$$\rho(x, \vartheta)|\lambda| \geqslant |\lambda| \, |\langle f, \vartheta \rangle| \geqslant \langle f, \lambda\vartheta \rangle \tag{25}$$

for each $\lambda \in \mathbb{R}$. Hence, the virtual work principle is satisfied.

In conclusion we obtain the expression

$$S = \big\{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q \; ; \; |\langle f, \vartheta \rangle| \leqslant \rho(x, \vartheta), \tau = 0\big\} \tag{26}$$

for the constitutive set of the system. ▲

Example 4. The present example gives a formal description of experiments performed by Coulomb in his study of static friction. Notational conventions of Example 1 will be used. Let a material point with configuration $q \in Q = M$ be constrained to the set

$$C^0 = \{q \in Q \; ; \; \langle g(k), q - q_0 \rangle \geqslant 0\} , \tag{27}$$

where q_0 is a point in Q and $k \in V$ is a unit vector. The boundary

$$\partial C^0 = \{q \in Q \; ; \; \langle g(k), q - q_0 \rangle = 0\} \tag{28}$$

is a plane passing through q_0 and orthogonal to k. In its displacements along the boundary the point encounters friction proportional to the component of the external force pressing the point against the boundary. The system is characterized by the virtual work function $\sigma = 0$ defined on the non holonomic constraint

$$\begin{aligned} C^1 = \Big\{ &(q, \delta q) \in Q \times \delta Q \; ; \; \langle g(k), q - q_0 \rangle \geqslant 0, \\ &\langle g(k), \delta q \rangle \geqslant \rho \sqrt{\|\delta q\|^2 - \langle g(k), \delta q \rangle^2} \text{ if } \; \langle g(k), q - q_0 \rangle = 0 \Big\}, \end{aligned} \tag{29}$$

where $\rho > 0$ is the coefficient of friction.

The principle of virtual work defines the constitutive set

$$S = \Big\{ (q, f) \in Q \times F \; ; \; \forall_{\delta q \in \in \delta Q} \; \text{ if } (q, \delta q) \in C^1, \text{ then } \langle f, \delta q \rangle \leqslant 0 \Big\}. \tag{30}$$

If the material point is not on the boundary, then $\langle g(k), q - q_0 \rangle > 0$. The virtual displacements are not restricted and $(q, f) \in S$ if and only if $f = 0$.

If the material point is on the boundary, then $\langle g(k), q - q_0 \rangle = 0$. We show that in this case $(q, f) \in S$ if and only if the inequality

$$\sqrt{\|f\|^2 - \langle f, k \rangle^2} + \rho \langle f, k \rangle \leqslant 0 \tag{31}$$

is satisfied. The inequality $\|f\|^2 - \langle f, k \rangle^2 \geqslant 0$ is guaranteed by the Schwarz inequality $|\langle f, k \rangle| \leqslant \|f\| \|k\|$.

Let $(q, f) \in S$ and let $\|f\|^2 - \langle f, k \rangle^2 = 0$. This will be the case when $f = -\|f\| g(k)$. The case $f = \|f\| g(k)$ with $\|f\| \neq 0$ is excluded by the virtual work principle. The inequality (31) is satisfied since

$$\sqrt{\|f\|^2 - \langle f, k \rangle^2} + \rho \langle f, k \rangle = -\rho \|f\| \leqslant 0. \tag{32}$$

We analyse the case of $(q, f) \in S$ and $\|f\|^2 - \langle f, k \rangle^2 > 0$. The virtual displacement $(q, \delta q)$ with

$$\delta q = g^{-1}(f) - \langle f, k \rangle k + \rho \sqrt{\|f\|^2 - \langle f, k \rangle^2} k \tag{33}$$

is in C^1 since

$$\langle g(k), \delta q \rangle = \rho \sqrt{\|f\|^2 - \langle f, k \rangle^2}. \tag{34}$$

From the principle of virtual work and from

$$\langle f, \delta q \rangle = \|f\|^2 - \langle f, k \rangle^2 + \rho \sqrt{\|f\|^2 - \langle f, k \rangle^2} \langle f, k \rangle \tag{35}$$

it follows that

$$\|f\|^2 - \langle f, k \rangle^2 + \rho \sqrt{\|f\|^2 - \langle f, k \rangle^2} \langle f, k \rangle \leqslant 0. \tag{36}$$

The inequality (31) is obtained since $\|f\|^2 - \langle f, k \rangle^2 > 0$.

Let the point be on the boundary and let the inequality (31) be satisfied. The Schwarz inequality

$$|\langle g(u), v \rangle - \langle g(k), u \rangle \langle g(k), v \rangle| \leqslant \sqrt{\|u\|^2 - \langle g(k), u \rangle^2} \sqrt{\|v\|^2 - \langle g(k), v \rangle^2} \tag{37}$$

for the bilinear symmetric form

$$(u, v) \mapsto \langle g(u), v \rangle - \langle g(k), u \rangle \langle g(k), v \rangle \tag{38}$$

applied to the pair $(g^{-1}(f), \delta q)$ leads to the inequality

$$\langle f, \delta q\rangle - \langle f, k\rangle\langle g(k), \delta q\rangle \leqslant \sqrt{\|f\|^2 - \langle f, k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2}. \tag{39}$$

If $\sqrt{\|f\|^2 - \langle f, k\rangle^2} + \rho\langle f, k\rangle \leqslant 0$, and $\langle g(k), \delta q\rangle \geqslant \rho\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2}$, then

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2} \leqslant -\rho\langle f, k\rangle\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2} \leqslant -\langle f, k\rangle\langle g(k), \delta q\rangle. \tag{40}$$

It follows that $\langle f, \delta q\rangle \leqslant 0$. Hence, (q, f) is in the constitutive set S.

We have shown that

$$\begin{aligned} S = \{(q, f) \in Q \times F \,;\, & \langle g(k), q - q_0\rangle \geqslant 0,\ f = 0 \ \text{ if } \ \langle g(k), q - q_0\rangle > 0 \\ & \text{and } \ \sqrt{\|f\|^2 - \langle f, k\rangle^2} + \rho\langle f, k\rangle \leqslant 0 \ \text{ if } \ \langle g(k), q - q_0\rangle = 0\Big\}. \end{aligned} \tag{41}$$

We will interpret the friction coefficient ρ in terms of the components $\delta q^{\perp} = \langle g(k), \delta q\rangle k$ and $\delta q^{\shortparallel} = \delta q - \langle g(k), \delta q\rangle k$ orthogonal and parallel to the surface ∂C^0. The inequality

$$\langle g(k), \delta q\rangle \geqslant \rho\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2} \tag{42}$$

translates into the inequality

$$\|\delta q^{\perp}\| \geqslant \rho\|\delta q^{\shortparallel}\| \tag{43}$$

for the norms

$$\|\delta q^{\perp}\| = \langle g(k), \delta q\rangle \quad \text{and} \quad \|\delta q^{\shortparallel}\| = \sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2}. \tag{44}$$

Let q be on the boundary of C^0 and let ϑ be the angle between a displacement δq and k. The pair $(q, \delta q)$ is in C^1 if $\vartheta = 0$ or $\cot\vartheta \geqslant \rho$. ▲

3. Convex sets and functions.

Let V be a vector space of finite dimension m. The *convex hull* of a subset $A \subset V$ is the set

$$\operatorname{cnv} A = \left\{ v \in V \,;\, \exists_{a_1, a_2, \dots, a_k \in A,\ \lambda_1, \lambda_2, \dots, \lambda_k \in [0,1]} \textstyle\sum_{i=1}^{k} = 1 \text{ and } v = \sum_{i=1}^{k} \lambda_i a_i \right\}. \tag{45}$$

The inclusion $A \subset \operatorname{cnv} A$ is always satisfied. The set A is said to be *convex* if $A = \operatorname{cnv} A$. The convex hull $\operatorname{cnv} A$ is the smallest convex subset of V containing A. The *affine hull* of A is the set

$$\operatorname{aff} A = \left\{ v \in V \,;\, \exists_{a_1, a_2, \dots, a_k \in A,\ \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}} \textstyle\sum_{i=1}^{k} = 1 \text{ and } v = \sum_{i=1}^{k} \lambda_i a_i \right\}. \tag{46}$$

It is the smallest affine subspace of V containing A.

Let $A \subset V$ be a convex subset. The set $\operatorname{aff} A$ is a topological space. The *relative interior* $\operatorname{rint} A$ of A is the topological interior of A in $\operatorname{aff} A$. It is the biggest subset of A open in $\operatorname{aff} A$.

Proposition 1. *The relative interior* $\operatorname{rint} A$ *of a non empty convex set* $A \subset V$ *is not empty.*

Let $\sigma : C \to \mathbb{R}$ be a function defined on a subset $C \subset V$. The *overgraph* of σ is the set

$$\operatorname{ovgr} \sigma = \{(v, r) \in V \times \mathbb{R} \,;\, v \in C, r \geqslant \sigma(v)\}. \tag{47}$$

The function σ is said to be *convex* if $\operatorname{ovgr}\sigma$ is a convex subset of $V \times \mathbb{R}$. The domain C of a convex function is convex. The function σ is said to be *concave* if the function $-\sigma$ is convex. A convex function σ is said to be *closed* if $\operatorname{ovgr}\sigma$ is closed in $V \times \mathbb{R}$. A concave function σ is said to be *closed* if the function $-\sigma$ is closed.

Given a static system characterized by constraint sets C^0 and C^1 and a virtual work function $\sigma\colon C^1 \to \mathbb{R}$ we introduce sets

$$C^1_q = \left\{\delta q \in \delta Q\ ;\ (q, \delta q) \in C^1\right\} \tag{48}$$

for each $q \in C^0$ and functions

$$\sigma_q\colon C^1_q \to \mathbb{R}\colon \delta q \mapsto \sigma(q, \delta q). \tag{49}$$

The static system is considered *convex* if the functions σ_q are convex. The three systems in the typical examples are convex. We list the objects S_q and σ_q in the three examples.

(1) Example 1:

$$C^1_q = \{\delta q \in \delta Q\ ;\ \langle g(q - q_0), \delta q\rangle = 0\}, \tag{50}$$

$$\sigma_q = 0. \tag{51}$$

(2) Example 2:

$$C^1_q = \delta Q, \tag{52}$$

$$\sigma_q\colon \delta Q \to\colon \delta q \mapsto \rho(q)\|\delta q\|. \tag{53}$$

(3) Example 3:

$$C^1_{(x,\vartheta)} = \left\{(\delta x, \delta\vartheta) \in V \times \mathsf{T}_\vartheta D\ ;\ \exists_{\lambda\in\mathbb{R}}\ \delta x = \lambda\vartheta\right\}, \tag{54}$$

$$\sigma_{(x,\vartheta)}\colon C^1_{(x,\vartheta)} \to \mathbb{R}\colon (\delta x, \delta\vartheta) \mapsto \rho(x)\|\delta x\|. \tag{55}$$

(4) Example 4:

$$C^1_q = \left\{\delta q \in \delta Q\ ;\ \langle g(k), \delta q\rangle \geqslant \rho\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2}\ \text{if}\ \ \langle g(k), q - q_0\rangle = 0\right\}, \tag{56}$$

$$\sigma_q = 0. \tag{57}$$

4. Separation of sets.

A non zero affine function $h\colon V \to \mathbb{R}$ is said to *separate* non empty subsets $A \subset V$ and $B \subset V$ if $h|A \geqslant 0$ and $h|B \leqslant 0$. The separation is said to be *strong* if $h|A > 0$ and $h|B < 0$. If a function h separates sets A and B, the level set $H = h^{-1}(0))$ of the function is said to *separate* the sets A and B. If the function h separates the sets A and B strongly, then the level set $H = h^{-1}(0))$ is said to *separate* the sets A and B *strongly*.

The following theorem is known as the Separation Theorem.

THEOREM 1. *Let A and B be convex subsets of V. If*

$$\operatorname{rint} A \cap \operatorname{rint} B = \emptyset, \tag{58}$$

then there is an affine function h separating the sets A and B.

THEOREM 2. *Let $A \subset V$ and $B \subset V$ be convex, closed and non empty. If A is compact and $A \cap B = \emptyset$, then there is an affine function h strongly separating the sets A and B.*

5. The Legendre transformation in a vector space, the homogeneous case.

Let

$$\sigma\colon C \to \mathbb{R} \tag{59}$$

be a function defined on a subset C of a vector space V. The set C is assumed to be a *cone* in the sense that if $v \in C$, then $kv \in C$ for each $k \geqslant 0$. The function σ is assumed to be positive homogeneous in the sense that $\sigma(kv) = k\sigma(v)$ for each $k \geqslant 0$. The set

$$S = \left\{f \in V^*\ ;\ \forall_{v\in C}\ \sigma(v) - \langle f, v\rangle \geqslant 0\right\} \tag{60}$$

is called the *Legendre transform* of σ.

Proposition 2. *The Legendre transform S of a positive homogeneous function $\sigma\colon C \to \mathbb{R}$ is convex and closed.*

Proof: If $f_1 \in S$ and $f_2 \in S$, then for each $v \in C$ and each $s \in [0,1]$ we have

$$\begin{aligned}\sigma(v) - \langle (1-s)f_1 + sf_2, v\rangle &= (1-s)\sigma(v) - \langle (1-s)f_1, v\rangle + s\sigma(v) - \langle sf_2, v\rangle \\ &= (1-s)(\sigma(v) - \langle f_1, v\rangle) + s(\sigma(v) - \langle f_2, v\rangle) \geqslant 0\end{aligned} \tag{61}$$

since $\sigma(v) - \langle f_1, v\rangle \geqslant 0$ and $\sigma(v) - \langle f_2, v\rangle \geqslant 0$. It follows that $(1-s)f_1 + sf_2 \in S$. Hence, S is convex. Continuity of the mapping $\langle\,\cdot\,, v\rangle\colon V^* \to \mathbb{R}\colon f \mapsto \langle f, v\rangle$ for each $v \in V$ implies that S is closed. ■

The constitutive sets S derived from the virtual work functions σ in the examples of static systems are obtained by applying of the Legendre transformation to functions σ_q defined in (49). The Legendre transforms S_q are then combined. The constitutive sets are the unions

$$\bigcup_{q \in Q} \{q\} \times S_q. \tag{62}$$

We list the Legendre transforms obtained in the three examples.

(1) Example 1:
$$S_q = \left\{ f \in F\ ;\ f = a^{-2}\langle f, q - q_0\rangle g(q - q_0) \right\}. \tag{63}$$

(2) Example 2:
$$S_q = \{ f \in F\ ;\ \|f\| \leqslant \rho(q)\}. \tag{64}$$

(3) Example 3:
$$S_{(x,\vartheta)} = \left\{ (f, \tau) \in V^* \times \mathsf{T}^*_\vartheta D\ ;\ |\langle f, \vartheta\rangle| \leqslant \rho(x), \tau = 0 \right\}. \tag{65}$$

(4) Example 4:
$$S_q = \left\{ f \in F\ ;\ \sqrt{\|f\|^2 - \langle f, k\rangle^2} + \rho\langle f, k\rangle \leqslant 0 \text{ if } \ \langle g(k), q - q_0\rangle = 0 \right\}, \tag{66}$$

6. The inverse Legendre transformation.

Let S be a subset of V^*. We introduce the set

$$C = \left\{ v \in V\ ;\ \sup_{f \in S}\langle f, v\rangle < \infty \right\} \tag{67}$$

and the function

$$\sigma\colon C \to \mathbb{R}\colon v \mapsto \sup_{f \in S}\langle f, v\rangle. \tag{68}$$

Note that the set C is a cone and σ is positive homogeneous.

Proposition 3. *The set C is convex. The function σ is convex and closed.*

Proof: For $v_1 \in C$, $v_2 \in C$, and $s \in [0,1]$ the combination $sv_1 + (1-s)v_2$ is in C since

$$\begin{aligned}\sup_{f \in S}\langle f, sv_1 + (1-s)v_2\rangle &\leqslant \sup_{f \in S} s\langle f, v_1\rangle + \sup_{f \in S}(1-s)\langle f, v_2\rangle \\ &= s\sup_{f \in S}\langle f, v_1\rangle + (1-s)\sup_{f \in S}\langle f, v_2\rangle < \infty.\end{aligned} \tag{69}$$

It follows that C is convex. Moreover,

$$\sigma(sv_1 + (1-s)v_2) \leqslant s\sigma(v_1) + (1-s)\sigma(v_2). \tag{70}$$

Hence, σ is convex.

Let (v, r) be the limit of a sequence (v_n, r_n) of elements of $\operatorname{ovgr}\sigma$. The relation

$$r_n \geqslant \sigma(v_n) = \sup_{f\in S}\langle f, v_n\rangle \tag{71}$$

is a consequence of the construction of σ and the definition of the overgraph. It follows from this relation that for each $\varepsilon > 0$ the inequalities

$$\langle f, v_n\rangle \leqslant r_n < r + \varepsilon \tag{72}$$

hold for all $f \in S$ and sufficiently large n. Convergence of $\langle f, v_n\rangle$ to $\langle f, v\rangle$ implies that $\langle f, v\rangle \leqslant r + \varepsilon$ for each $f \in S$ and each $\varepsilon > 0$. Hence, $\langle f, v\rangle \leqslant r$ and $v \in C$. Moreover

$$\sup_{f\in S}\langle f, v\rangle \leqslant r + \varepsilon \tag{73}$$

for each $f \in S$ and each $\varepsilon > 0$. Hence, $\sigma(v) \leqslant r$ and $(v, r) \in \operatorname{ovgr}\sigma$. We have shown that σ is closed since $\operatorname{ovgr}\sigma$ is closed. ■

Rockafellar [2] calls the function σ the *support function* for the set S. The following theorem is based on the results established in [2].

THEOREM 3. *The Legendre transformation and the inverse Legendre transformation establish a one to one correspondence between positive homogeneous closed convex functions defined on cones in V and non empty closed convex subsets of V^*.*

It follows from Theorem 3 that the constitutive set provides a complete characterization of a convex static system. Systems presented in the three examples above are convex. Each of the systems is equally well characterized by the virtual work function σ or by the constitutive set S.

7. Generating families and the principle of virtual work for partially controlled systems.

There are two configuration spaces involved in the description of a partially controlled system. There is the *internal configuration space* $\overline{Q}$ and the *control configuration space* Q. The two spaces are components of a differential fibration

$$\begin{tikzcd} \overline{Q} \arrow[d, "\eta"'] \\ Q \end{tikzcd} \quad . \tag{74}$$

There is the *internal energy function*

$$\overline{U} : \overline{Q} \to \mathbb{R} \tag{75}$$

interpreted as a *family* of functions defined on fibres of the fibration η. The symbol $(\overline{U}, \eta)$ is used to denote this family.

A generating family $(\overline{U}, \eta)$ generates the constitutive set

$$S = \left\{ f \in \mathsf{T}^*Q \; ; \; \exists_{\overline{q}\in\overline{Q}} \; \eta(\overline{q}) = \pi_Q(f) \;\; \forall_{\delta\overline{q}\in\mathsf{T}_{\overline{q}}\overline{Q}} \; \langle \mathrm{d}\overline{U}, \delta\overline{q}\rangle = \langle f, \, \mathsf{T}\eta(\delta\overline{q})\rangle \right\} \tag{76}$$

of a partially controlled system.

We denote by $\mathsf{V}\overline{Q}$ the subbundle

$$\left\{ \delta\overline{q} \in \mathsf{T}\overline{Q} \; ; \; \mathsf{T}\eta(\delta\overline{q}) = 0 \right\} \tag{77}$$

of vertical vectors. The set

$$Cr(\overline{U},\eta) = \left\{\overline{q} \in \overline{Q}\ ;\ \langle \mathrm{d}\overline{U}, \delta\overline{q}\rangle = 0 \ \text{ for each } \ \delta\overline{q} \in \mathsf{V}_{\overline{q}}\overline{Q}\right\} \tag{78}$$

is called the *critical set* of the family. If $\overline{q}$ satisfies the conditions stated in the definition of S, then the equality $\langle \mathrm{d}\overline{U}(\overline{q}), \delta\overline{q}\rangle = 0$ is obtained with $\delta q = 0$ and any vertical vector $\delta\overline{q} \in \overline{Q}_{\overline{q}}$. It follows that $\overline{q} \in Cr(\overline{U},\eta)$.

There is a mapping

$$\kappa(\overline{U},\eta) \colon Cr(\overline{U},\eta) \to \mathsf{T}^*Q \tag{79}$$

characterized by

$$\langle \kappa(\overline{U},\eta)(\overline{q}), \delta q\rangle = \langle \mathrm{d}\overline{U}, \delta\overline{q}\rangle \tag{80}$$

for each $\delta q \in \mathsf{T}_{\eta(\overline{q})}Q$ and each $\delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}$ such that $\mathsf{T}\eta(\delta\overline{q}) = \delta q$. The constitutive set is the image of $\kappa(\overline{U},\eta)$.

Example 5. A material point with configuration q' in the affine space $Q = M$ is connected to a fixed point q_0 with a rigid rod of length a. The model space V of Q will be denoted by δQ and the dual space V^* will be denoted by F. The set

$$D = \{\vartheta \in V\ ;\ \langle g(\vartheta), \vartheta\rangle = 1\} \tag{81}$$

will be used as the configuration space of the point q'. Its actual configuration of q' in Q is given by $q' = q_0 + a\vartheta$. A second material point with configuration q is tied elastically to q' with a spring of spring constant k. The configuration space is the product $\overline{Q} = Q \times D$. The function

$$\overline{U} \colon \overline{Q} \to \mathbb{R} \colon (q,\vartheta) \mapsto \frac{k}{2}\|q - (q_0 + a\vartheta)\|^2 \tag{82}$$

is the internal energy. Spaces $\mathsf{T}D$ and T^*D are defined as in Example 3. Virtual displacements are elements of the space $\mathsf{T}\overline{Q} = Q \times \delta Q \times \mathsf{T}D$. The constitutive set is a subset of the space $\mathsf{T}^*\overline{Q} = Q \times F \times \mathsf{T}^*D$. The set

$$\overline{S} = \left\{(q,f,\vartheta,\tau) \in \mathsf{T}^*\overline{Q}\ ;\ f = kg(q - (q_0 + a\vartheta)),\ \tau = -ka\left(g(q-q_0) - \langle g(q-q_0),\vartheta\rangle g(\vartheta)\right)\right\} \tag{83}$$

is the constitutive set.

If the configuration ϑ is not controlled, then $\tau = 0$ and we have an example of a partially controlled system. The space $\overline{Q}$ is the internal configuration space and the control configuration space is the space $Q = M$. The canonical projection

$$\eta \colon \overline{Q} \to Q \colon (q,\vartheta) \mapsto q \tag{84}$$

is the relation between the two spaces $\overline{Q}$ and Q. The internal energy defines a family $(\overline{U},\eta)$. The set

$$Cr(\overline{U},\eta) = \left\{(q,\vartheta) \in \overline{Q}\ ;\ \vartheta \in D \ \text{ if } \ q = q_0,\ \ \vartheta = \pm(q-q_0)\|q-q_0\|^{-1}\right\}. \tag{85}$$

is the critical set. The description

$$\kappa(\overline{U},\eta)(q,\vartheta) = \begin{cases} -kag(\vartheta) & \text{if } \ q = q_0 \text{ and } \vartheta \in D \\ kg(q-q_0)\left(1 - a\|q-q_0\|^{-1}\right) & \text{if } \ q \neq q_0 \text{ and } \vartheta = +(q-q_0)\|q-q_0\|^{-1} \\ kg(q-q_0)\left(1 + a\|q-q_0\|^{-1}\right) & \text{if } \ q \neq q_0 \text{ and } \vartheta = -(q-q_0)\|q-q_0\|^{-1} \end{cases} \tag{86}$$

of the mapping $\kappa(\overline{U},\eta) \colon Cr(\overline{U},\eta) \to \mathsf{T}^*Q = Q \times F$ is derived from the general expression

$$\kappa(\overline{U},\eta)(q,\vartheta) = kg(q - (q_0 + a\vartheta)). \tag{87}$$

The set

$$S = \big\{(q,f) \in \mathsf{T}^*Q \; ; \; \|f\| = ka \;\text{ if }\; q = q_0, \\ f = k\left(1 \pm a\|q-q_0\|^{-1}\right) g(q-q_0) \;\text{ if }\; q \neq q_0\big\} \tag{88}$$

is the constitutive set. It is the image of $\kappa(\overline{U},\eta)$.

Note that the critical set is not the image of a section of η. For each control configuration q we have two different internal equilibrium configurations (q,ϑ) if $q \neq q_0$ and an infinity of internal equilibrium configurations if $q = q_0$. The external force necessary to maintain the control configuration q depends on the internal configuration. Thus even if the internal configuration is not directly observed its presence can not be ignored. ▲

8. Reduction of generating families.

Let $(\overline{U},\eta)$ be a family generating the set (76). We have the following obvious proposition.

PROPOSITION 4. *Let $\overline{q} \in Cr(\overline{U},\eta)$. The single point set*

$$S_{\overline{q}} = \left\{ f \in \mathsf{T}^*Q \; ; \; \pi_Q(f) = \eta(\overline{q}) \;\; \forall_{\delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}} \; \mathrm{d}\overline{U}(\delta\overline{q}) = \langle f, \, \mathsf{T}\eta(\delta\overline{q})\rangle \right\}. \tag{89}$$

is represented in the form

$$S_{\overline{q}} = \left\{ f \in \mathsf{T}^*Q \; ; \; \pi_Q(f) = \eta(\overline{q}) \;\; \forall_{\delta q \in \mathsf{T}_{\eta(\overline{q})}Q} \; \sigma_{\overline{q}}(\delta q) = \langle f, \, \delta q\rangle \right\}, \tag{90}$$

where

$$\sigma_{\overline{q}} \colon \mathsf{T}_{\eta(\overline{q})}Q \to \mathbb{R} \colon \delta q \mapsto \mathrm{d}\overline{U}(\delta\overline{q}), \;\; \delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}, \;\; \mathsf{T}\eta(\delta\overline{q}) = \delta q. \tag{91}$$

It follows from the above proposition that if $Cr(\overline{U},\eta)$ is the image of a section $\zeta \colon Q \to \overline{Q}$ of the fibration η then the family $(\overline{U},\eta)$ generating the set S in (76) can be replaced by the function

$$\sigma \colon \mathsf{T}Q \to \mathbb{R} \colon (\delta q) \mapsto \sigma_{\zeta(\tau_Q(\delta q))}(\delta q), \tag{92}$$

where $\sigma_{\zeta(\tau_Q(\delta q))}$ is the function $\sigma_{\overline{q}}$ defined in the the formula (91) with $\overline{q} = \zeta(\tau_Q(\delta q))$. It is obvious that $\sigma = \mathrm{d}(\overline{U} \circ \zeta)$. Thus the set S is generated by the function $U = \overline{U} \circ \zeta$.

EXAMPLE 6. Three material points with configurations q_0, q, and q' in the affine space M are interconnected with springs with spring constants k, k', and k''. The point q_0 is fixed and not controlled. The two points q and q' are not constrained. The configuration q' is not controlled. The internal configuration space is the affine space $\overline{Q} = M \times M$ of internal configurations $\overline{q} = (q,q')$ modelled on $\delta\overline{Q} = V \oplus V$. The control configuration space is the space $Q = M$ of controlled configurations q and δQ is the model space. The dual of δQ will be denoted by F. The canonical projection

$$\eta \colon \overline{Q} \to Q \colon \overline{q} = (q,q') \mapsto q \tag{93}$$

is the relation between the two spaces. The internal energy is the function

$$\overline{U} \colon \overline{Q} \to \mathbb{R} \colon \overline{q} = (q,q') \mapsto \frac{k}{2}\|q-q_0\|^2 + \frac{k'}{2}\|q'-q_0\|^2 + \frac{k''}{2}\|q'-q\|^2. \tag{94}$$

The internal energy defines a a family $(\overline{U},\eta)$ of functions on fibres of the projection η. The critical set

$$Cr(\overline{U},\eta) = \big\{\overline{q} = (q,q') \in \overline{Q} \; ; \; k'g(q'-q_0) + k''g(q'-q) = 0\big\} \tag{95}$$

of the family is the image of the section

$$\zeta : Q \to \overline{Q} : q = q \mapsto (q, q_0 + k''(k' + k'')^{-1}(q - q_0)) \tag{96}$$

of the projection η.

The constitutive set is the set

$$S = \left\{ (q, f) \in Q \times F \ ; \ f = \frac{kk' + kk'' + k'k''}{k' + k''} \, g(q - q_0) \right\}. \tag{97}$$

Note that the presence of the material point with configuration q' can be ignored. This is due to the fact that the critical set is the image of a section of the projection η. The constitutive set is generated by the reduced internal energy function

$$U = \overline{U} \circ \zeta : Q \to \mathbb{R} : q \mapsto \frac{1}{2} \frac{kk' + kk'' + k'k''}{k' + k''} \|q - q_0\|^2. \tag{98}$$

▲

9. Generating families of forms.

A *generating family* consists of a differential fibration

$$\begin{array}{c} \overline{Q} \\ \eta \Big\downarrow \\ Q \end{array} \tag{99}$$

and a form $\overline{\sigma} : \mathsf{T}\,\overline{Q} \to \mathbb{R}$. We refer to functions on $\mathsf{T}\overline{Q}$ as *forms* in order to distinguish such functions from functions on $\overline{Q}$. We do not imply that the forms are linear. The form $\overline{\sigma}$ is differentiable on $\mathsf{T}\,\overline{Q}$ with the image of the zero section removed. It is positive homogeneous and convex on fibres of the tangent fibration $\tau_{\overline{Q}} : \mathsf{T}\,\overline{Q} \to \overline{Q}$. We denote by $\mathsf{V}\,\overline{Q}$ the subbundle

$$\left\{ \delta\overline{q} \in \mathsf{T}\,\overline{Q} \ ; \ \mathsf{T}\eta(\delta\overline{q}) = 0 \right\} \tag{100}$$

of vertical vectors. The form $\overline{\sigma}$ defines a family $(\overline{\sigma}, \eta)$ of forms $\overline{\sigma}_q$ associated with fibres of the fibration η. Each form $\overline{\sigma}_q$ is the restriction of the form $\overline{\sigma}$ to the set

$$\{ \delta\overline{q} \in \mathsf{T}\,\overline{Q} \ ; \ \eta(\tau_{\overline{Q}}(\delta\overline{q})) = q \}. \tag{101}$$

The set

$$Cr(\overline{\sigma}, \eta) = \left\{ \overline{q} \in \overline{Q} \ ; \ \overline{\sigma}(\delta\overline{q}) \geqslant 0 \ \text{ for each } \ \delta\overline{q} \in \mathsf{V}_{\overline{q}}\overline{Q} \right\} \tag{102}$$

is called the *critical set* of the family.

A generating family $(\overline{\sigma}, \eta)$ generates the set

$$\begin{aligned} S = \Big\{ f \in \mathsf{T}^*Q \ ; \ q = \pi_Q(f) \in Q, \exists_{\overline{q} \in \overline{Q}_q} \ \text{ if } \delta q \in \mathsf{T}_q Q, \\ \delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}, \ \text{and } \mathsf{T}\eta(\delta\overline{q}) = \delta q, \ \text{then } \overline{\sigma}(\delta\overline{q}) \geqslant \langle f, \, \delta q \rangle \Big\}. \end{aligned} \tag{103}$$

If $\overline{q}$ satisfies the conditions stated in the definition of S, then the inequality $\overline{\sigma}(\delta\overline{q}) \geqslant 0$ is obtained with $\delta q = 0$ and any vertical vector $\delta\overline{q} \in \mathsf{V}_{\overline{q}}\overline{Q}$. It follows that $\overline{q} \in Cr(\overline{\sigma}, \eta)$. Consequently,

$$S = \bigcup_{\overline{q} \in Cr(\overline{\sigma}, \eta)} S_{\overline{q}} \, , \tag{104}$$

where

$$\begin{aligned} S_{\overline{q}} = \big\{ f \in \mathsf{T}^*Q \ ; \ q = \pi_Q(f) = \eta(\overline{q}), \ \text{if } \delta q \in \mathsf{T}_q Q, \delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q} \\ \text{and } \mathsf{T}\eta(\delta\overline{q}) = \delta q, \ \text{then } \overline{\sigma}(\delta\overline{q}) \geqslant \langle f, \, \delta q \rangle \big\} \, . \end{aligned} \tag{105}$$

PROPOSITION 5. *Let $\overline{q} \in Cr(\overline{\sigma}, \eta)$. The set $S_{\overline{q}}$ is not empty.*

Proof: It follows from the Separation Theorem that there is a hyperplane in $\mathsf{T}_{\overline{q}}\overline{Q} \times \mathbb{R}$ separating the subspace $\mathsf{V}_{\overline{q}}\overline{Q} \times \{0\}$ and the overgraph $\operatorname{ovgr}\overline{\sigma}_{\overline{q}}$ of $\overline{\sigma}_{\overline{q}} = \overline{\sigma}|\mathsf{T}_{\overline{q}}\overline{Q}$. This hyperplane is the graph of a linear function h on $\mathsf{T}_{\overline{q}}\overline{Q}$. Note that $\overline{\sigma}(\delta\overline{q}) \geqslant h(\delta\overline{q})$ and $h(\delta\overline{q}') = 0$ if $\delta\overline{q}' \in \mathsf{V}_{\overline{q}}\overline{Q}$. It follows that the function

$$f\colon \mathsf{T}_q Q \to \mathbb{R}\colon \delta q \mapsto h(\delta\overline{q}),\ \ \delta q = \mathsf{T}\eta(\delta\overline{q}) \tag{106}$$

is well defined, and $f \in S_{\overline{q}}$. ■

The relation

$$\kappa(\overline{\sigma},\eta)\colon Cr(\overline{\sigma},\eta) \to \mathsf{T}^*Q \tag{107}$$

defined by

$$\operatorname{graph}\kappa(\overline{\sigma},\eta) = \left\{(\overline{q}, f) \in Cr(\overline{\sigma},\eta) \times \mathsf{T}^*Q\ ;\ f = S_{\overline{q}}\right\} \tag{108}$$

generalizes the mapping $\kappa(\overline{U},\eta)$ introduced in Section 7. We will refer to the set $S_{\overline{q}}$ as the *contribution* to the constitutive set S from the critical point $\overline{q}$.

Example 7. A material point with configuration q in the affine space $Q = M$ is tied to a point $q' \in M$ with a spring of spring constant k. The point q' is subject to friction and left free. The internal configuration space is the affine space $\overline{Q} = M \times M$ of internal configurations $\overline{q} = (q, q')$ modelled on $\delta\,\overline{Q} = V \oplus V$. The control configuration space is the space $Q = M$ of controlled configurations q and δQ is the model space. The dual of δQ will be denoted by F. The canonical projection

$$\eta : \overline{Q} \to Q : (q, q') \mapsto q \tag{109}$$

is the relation between the two configuration spaces. The virtual work is the form

$$\overline{\sigma} : \overline{Q} \times \delta\,\overline{Q} \to \mathbb{R} : (q, q', \delta q, \delta q') \mapsto k\langle g(q'-q), \delta q' - \delta q\rangle + \rho\|\delta q'\| \tag{110}$$

This form together with the projection η define a generating family for the constitutive set. The set

$$\begin{aligned} Cr(\overline{\sigma},\eta) &= \left\{\overline{q} = (q,q') \in \overline{Q}\ ;\ \forall_{\delta q' \in V}\ k\langle g(q'-q), \delta q'\rangle + \rho\|\delta q'\| \geqslant 0\right\} \\ &= \left\{\overline{q} = (q,q') \in \overline{Q}\ ;\ \rho \geqslant k\|q'-q\|\right\} \end{aligned} \tag{111}$$

is the critical set.

The contribution $S_{(q,q')}$ from the critical point (q,q') to the constitutive set S is the set

$$\begin{aligned} S_{(q,q')} &= \left\{(q,f) \in Q \times F\ ;\ \forall_{\delta q \in \delta Q}\ \langle f, \delta q\rangle \leqslant \inf_{\delta q' \in V} \sigma(q,q',\delta q,\delta q')\right\} \\ &= \left\{(q,f) \in Q \times F\ ;\ \forall_{\delta q \in \delta Q}\ \langle f, \delta q\rangle \leqslant k\langle g(q-q'), \delta q\rangle\right\} \\ &= \left\{(q,f) \in Q \times F\ ;\ f = kg(q-q')\right\}, \end{aligned} \tag{112}$$

since

$$\inf_{\delta q' \in V} \sigma(q,q',\delta q,\delta q') = \inf_{\delta q' \in V} \left(k\langle g(q'-q), \delta q' - \delta q\rangle + \rho\|\delta q'\|\right) = -k\langle g(q'-q), \delta q\rangle \tag{113}$$

due to

$$k\langle g(q'-q), \delta q'\rangle + \rho\|\delta q'\| \geqslant 0. \tag{114}$$

The expression

$$S = \bigcup_{(q,q') \in Cr(\overline{\sigma},\eta)} S_{(q,q')} = \left\{(q,f) \in Q \times F\ ;\ \|f\| \leqslant \rho\right\} \tag{115}$$

is obtained for the constitutive set. Neither the configuration q' nor the spring constant k appear in this expression. For each configuration $q \in Q$ the configuration q' is in the set

$$\left\{q' \in M\ ;\ (q,q') \in Cr(\overline{\sigma},\eta)\right\} = \left\{q' \in M\ ;\ k\|q'-q\| \leqslant \rho\right\}, \tag{116}$$

▲

Example 8. A point with configuration q in $Q = M$ is tied to a point with configuration $q' \in M$ with a spring of spring constant k. The point q' is in turn tied to a fixed point $q_0 \in M$ with a spring of spring constant k'. The point q' is subject to friction and left free. We use the definitions and symbols introduced in Example 7. The virtual work function of the system is the form

$$\overline{\sigma} : \overline{Q} \times \delta\,\overline{Q} \to \mathbb{R} : (q, q', \delta q, \delta q') \mapsto k'\langle g(q' - q_0), \delta q'\rangle + k\langle g(q - q'), \delta q - \delta q'\rangle + \rho\|\delta q'\|. \tag{117}$$

It defines a generating family for the costitutive set together with the projection η. The set

$$\begin{aligned} Cr(\overline{\sigma}, \eta) &= \Big\{\overline{q} = (q, q') \in \overline{Q}\ ;\ \forall_{\delta q' \in V}\ k'\langle g(q' - q_0), \delta q'\rangle + k\langle g(q' - q), \delta q'\rangle + \rho\|\delta q'\| \geqslant 0\Big\} \\ &= \big\{\overline{q} = (q, q') \in \overline{Q}\ ;\ \|k' g(q' - q_0) + k g(q' - q)\| \leqslant \rho\big\} \end{aligned} \tag{118}$$

is the critical set.

If (q, q') is a critical point, then

$$\inf_{\delta q'} \sigma(q, q', \delta q, \delta q') = k\langle g(q - q'), \delta q\rangle \tag{119}$$

due to

$$k'\langle g(q' - q_0), \delta q'\rangle + k\langle g(q' - q), \delta q\rangle + \rho\|\delta q'\| \geqslant 0, \tag{120}$$

and the contribution $S_{(q,q')}$ to the constitutive set consists of a single covector $f = kg(q - q')$. The inequality

$$\|k' g(q' - q_0) + k g(q' - q)\| \leqslant \rho \tag{121}$$

is equivalent to

$$\left\| k g(q - q') - \frac{kk'}{k + k'} g(q - q_0) \right\| \leqslant \frac{k\rho}{k + k'}. \tag{122}$$

It follows that

$$S = \left\{(q, f) \in Q \times F\ ;\ \left\| f - \frac{kk'}{k + k'} g(q - q_0) \right\| \leqslant \frac{k\rho}{k + k'} \right\} \tag{123}$$

is the constitutive set. ▲

Example 9. We return to the static system of Example 5 modifying the system by subjecting the point with configuration q' to isotropic homogeneous static friction. The function

$$\overline{\sigma} : \overline{Q} \times \delta\overline{Q} \to \mathbb{R} : (q, \vartheta, \delta q, \delta\vartheta) \mapsto k\langle g(q - (q_0 + a\vartheta)), \delta q\rangle - ka\langle g(q - (q_0 + a\vartheta)), \delta\vartheta\rangle + \rho a\|\delta\vartheta\| \tag{124}$$

together with the projection ϑ define a generating family of forms. The critical set of the family is the set

$$Cr(\overline{\sigma}, \eta) = \big\{(q, \vartheta) \in \overline{Q}\ ;\ \vartheta \in D,\ \ \forall_{\delta\vartheta \in \mathsf{T}D}\ - ka\langle g(q - q_0), \delta\vartheta\rangle + \rho a\|\delta\vartheta\| \geqslant 0\big\}. \tag{125}$$

The inequality

$$k\|q - q_0 - \langle g(q - q_0), \vartheta\rangle\vartheta\| = k\sqrt{\|q - q_0\|^2 - \langle g(q - q_0), \vartheta\rangle^2} \leqslant \rho \tag{126}$$

is derived from the inequality

$$-ka\langle g(q' - q_0) - \langle g(q - q_0), \vartheta\rangle g(\vartheta), \delta\vartheta\rangle + \rho a\|\delta\vartheta\| = -ka\langle g(q' - q_0), \delta\vartheta\rangle + \rho a\|\delta\vartheta\| \geqslant 0 \tag{127}$$

by setting

$$\delta\vartheta = q - q_0 - \langle g(q - q_0), \vartheta\rangle\vartheta. \tag{128}$$

If

$$k\|q - q_0 - \langle g(q - q_0), \vartheta\rangle\vartheta\| \leqslant \rho, \tag{129}$$

then the inequality (127) follows from the Schwarz inequality

$$k\langle g(q-q_0)-\langle g(q-q_0),\vartheta\rangle g(\vartheta),\delta\vartheta\rangle \leqslant k\|q-q_0-\langle g(q-q_0),\vartheta\rangle\vartheta\|\|\delta\vartheta\|. \tag{130}$$

We have shown that

$$Cr(\overline{\sigma},\eta)=\left\{(q,\vartheta)\in\overline{Q}\ ;\ \vartheta\in D,\ \ k\sqrt{\|q-q_0\|^2-\langle g(q-q_0),\vartheta\rangle^2}\leqslant\rho\right\}. \tag{131}$$

The contribution to the constitutive set from $(q,\vartheta)\in Cr(\overline{\sigma},\eta)$ is the single covector

$$f=kg(q-(q_0-a\vartheta)). \tag{132}$$

The set

$$S=\left\{(q,f)\in Q\times F\ ;\ \exists_{\vartheta\in D}\ k\sqrt{\|q-q_0\|^2-\langle g(q-q_0),\vartheta\rangle^2}\leqslant\rho\,,\ f=kg(q-(q_0-a\vartheta))\right\} \tag{133}$$

is the constitutive set of the system.

▲

10. Reduction of generating families.

PROPOSITION 6. *Let $\sigma\colon A\to\mathbb{R}$ be a convex function defined on a convex subset A of a vector space V. Let W be a subspace of V, let*

$$pr\colon V\to V/W\colon v\mapsto[v] \tag{134}$$

be the canonical projection, and let $B=pr(A)$. The function

$$\rho\colon B\to\mathbb{R}\colon[v]\mapsto\inf_{w\in W}\sigma(v+w) \tag{135}$$

is convex if well defined in the sense that the defining formula (135) *assigns to the function finite values.*

PROOF: Suppose that ρ is well defined. Let $[v],[v']\in E/V$ and $0\leqslant s\leqslant 1$ For each $\varepsilon>0$ there exist vectors $v_\varepsilon\in[v], v'_\varepsilon\in[v']$ such that $\sigma(v_\varepsilon)\leqslant\rho([v])+\varepsilon$ and $\sigma(v'_\varepsilon)\leqslant\rho([v'])+\varepsilon$. We have the inequalities

$$\begin{aligned}\varepsilon+s\rho([v])+(1-s)\rho([v'])&\geqslant s\sigma(v_\varepsilon)+(1-s)\sigma(v'_\varepsilon)\\&\geqslant\sigma(sv_\varepsilon+(1-s)v'_\varepsilon)\\&\geqslant\rho(s[v]+(1-s)[v'])\end{aligned} \tag{136}$$

for each $\varepsilon>0$. It follows that

$$s\rho([v])+(1-s)\rho([v']\geqslant\rho(s[v]+(1-s)[v']). \tag{137}$$

Hence, ρ is convex ■

Similarly, for a concave function $\sigma\colon V\to\mathbb{R}$, the function

$$\rho\colon V/W\to\mathbb{R}\colon[v]\mapsto\sup_{w\in W}\sigma(v+w) \tag{138}$$

is concave.

If the set A is a cone, the set B is a cone as well. If σ is positive homogeneous, then ρ is positive homogeneous. If the function σ is convex and closed, it may happen that the function ρ is not closed.

Let $\eta\colon\overline{Q}\to Q$ be a differential fibration and let $(\overline{\sigma},\eta)$ be a generating family.

Proposition 7. *If $\overline{q} \in Cr(\overline{\sigma}, \eta)$, then the set*

$$S_{\overline{q}} = \left\{ f \in \mathsf{T}^*Q \; ; \; \pi_Q(f) = \eta(\overline{q}) \;\; \forall_{\delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}} \; \overline{\sigma}(\delta\overline{q}) \geqslant \langle f, \, \mathsf{T}\eta(\delta\overline{q})\rangle \right\} \tag{139}$$

is represented in the form

$$S_{\overline{q}} = \left\{ f \in \mathsf{T}^*Q \; ; \; \pi_Q(f) = \eta(\overline{q}) \;\; \forall_{\delta q \in \mathsf{T}_{\eta(\overline{q})}Q} \; \sigma_{\overline{q}}(\delta q) \geqslant \langle f, \, \delta q\rangle \right\}, \tag{140}$$

where

$$\sigma_{\overline{q}} \colon \mathsf{T}_{\eta(\overline{q})}Q \to \mathbb{R} \colon \delta q \mapsto \inf_{\delta\overline{q}} \overline{\sigma}(\delta\overline{q}), \;\; \delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}, \;\; \mathsf{T}\eta(\delta\overline{q}) = \delta q. \tag{141}$$

Proof: We show that the function $\sigma_{\overline{q}}$ is well defined. As in Proposition 5 we have the graph of a linear function h on $\mathsf{T}_{\overline{q}}\overline{Q}$ separating the subspace $\mathsf{V}_{\overline{q}}\overline{Q} \times \{0\}$ and the overgraph $\operatorname{ovgr} \overline{\sigma}_{\overline{q}}$ of $\overline{\sigma}_{\overline{q}} = \overline{\sigma}|\mathsf{T}_{\overline{q}}\overline{Q}$. It follows from the Separation Theorem that there is a hyperplane in $\mathsf{T}_{\overline{q}}\overline{Q} \times \mathbb{R}$ separating the subspace $\mathsf{V}_{\overline{q}}\overline{Q} \times \{0\}$ and the overgraph $\operatorname{ovgr} \overline{\sigma}_{\overline{q}}$ of $\overline{\sigma}_{\overline{q}} = \overline{\sigma}|\mathsf{T}_{\overline{q}}\overline{Q}$. This hyperplane is the graph of a linear function h on $\mathsf{T}_{\overline{q}}\overline{Q}$. Note that $\overline{\sigma}(\delta\overline{q}) \geqslant h(\delta\overline{q})$ and $h(\delta\overline{q}') = 0$ if $\delta\overline{q}' \in \mathsf{V}_{\overline{q}}\overline{Q}$. It follows that $\sigma_{\overline{q}}$ is well defined since

$$\inf_{\delta\overline{q}' \in \mathsf{V}_{\overline{q}}\overline{Q}} \overline{\sigma}(\delta\overline{q} + \delta\overline{q}') \geqslant h(\delta\overline{q}). \tag{142}$$

If $\delta q \in \mathsf{T}_{\eta(\overline{q})}Q$ and $f \in \mathsf{T}^*_{\eta(\overline{q})}Q$, the inequality

$$\overline{\sigma}_{\overline{q}}(\delta\overline{q}) \geqslant \langle f, \delta q\rangle \tag{143}$$

for each $\delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}$ such that $\mathsf{T}\eta(\delta\overline{q}) = \delta q$ is equivalent to

$$\inf_{\delta\overline{q}} \overline{\sigma}(\delta\overline{q}) \geqslant \langle f, \delta q\rangle, \;\; \delta\overline{q} \in \mathsf{T}_{\overline{q}}\overline{Q}, \;\; \mathsf{T}\eta(\delta\overline{q}) = \delta q. \tag{144}$$

■

It follows from Proposition 7 that if $Cr(\overline{\sigma}, \eta)$ is the image of a section $\zeta \colon Q \to \overline{Q}$ of the fibration η then the family $(\overline{\sigma}, \eta)$ generating the set (103) can be replaced by the generating form

$$\sigma \colon \mathsf{T}Q \to \mathbb{R} \colon (\delta q) \mapsto \sigma_{\zeta(\tau_Q(\delta q))}(\delta q), \tag{145}$$

where $\sigma_{\zeta(\tau_Q(\delta q))}$ is the form $\sigma_{\overline{q}}$ defined in the the formula (141) with $\overline{q} = \zeta(\tau_Q(\delta q))$. The form σ is a legitimate generating form since on fibres of the tangent fibration τ_Q it is convex as a consequence of Proposition 6 and also positive homogeneous.

Example 10. A point with configuration q' is tied to a fixed point q_0 with a spring of spring constant k'. A second point with configuration q is tied to q' with a spring of spring constant k. The point q' is left free and the point q is subject to friction. As in Example 7, the internal configuration space is the affine space $\overline{Q} = M \times M$ of internal configurations $\overline{q} = (q, q')$ modelled on $\delta\,\overline{Q} = V \oplus V$. The control configuration space is the space $Q = M$ of controlled configurations q and δQ is the model space. The dual of δQ is denoted by F. The canonical projection

$$\eta \colon \overline{Q} \to Q \colon \overline{q} = (q, q') \mapsto q \tag{146}$$

is the relation between the two spaces. The virtual work function of the system is the form

$$\overline{\sigma} \colon \overline{Q} \times \delta\,\overline{Q} \to \mathbb{R} \colon (q, q', \delta q, \delta q') \mapsto k'\langle g(q' - q_0), \delta q'\rangle + k\langle g(q - q'), \delta q - \delta q'\rangle + \rho\sqrt{\langle g(\delta q), \delta q\rangle}. \tag{147}$$

Together with the projection η it defines a generating family $(\overline{\sigma}, \eta)$ for the costitutive set. The critical set

$$\begin{aligned} Cr(\overline{\sigma}, \eta) &= \Big\{ \overline{q} = (q, q') \in \overline{Q} \; ; \; \forall_{\delta q' \in V} \; k' \langle g(q' - q_0), \delta q' \rangle + k \langle g(q' - q), \delta q' \rangle \geqslant 0 \Big\} \\ &= \big\{ \overline{q} = (q, q') \in \overline{Q} \; ; \; k' g(q' - q_0) + k g(q' - q) = 0 \big\} \end{aligned} \tag{148}$$

is the image of the section

$$\zeta : Q \to \overline{Q} : q \mapsto (q, q_0 + \frac{k}{k' + k}(q - q_0)) \tag{149}$$

of η. It follows that the family $(\overline{\sigma}, \eta)$ can be reduced to the generating form

$$\sigma : Q \times \delta Q \to \mathbb{R} : (q, \delta q) \mapsto \frac{k' k}{k' + k} \langle g(q - q_0), \delta q \rangle + \rho \| \delta q \|. \tag{150}$$

The constitutive set is the set

$$\begin{aligned} S &= \left\{ (q, f) \in Q \times F \; ; \; \frac{k' k}{k' + k} \langle g(q - q_0) - f, \delta q \rangle + \rho \| \delta q \| \geqslant 0 \right\} \\ &= \left\{ (q, f) \in Q \times F \; ; \; \left\| \frac{k' k}{k' + k} g(q - q_0) - f \right\| \leqslant \rho \right\}. \end{aligned} \tag{151}$$

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